κ -deformation from 2+1 gravity

C&TCfQG – Roma, September 2014

The plan

- Kappa-deformation
- Kappa-Carroll particle
- Comments

JKG & T. Trześniewski, PLB 737, 267, ArXiv 1408.0154

Why deformations?

- It is possible that in some regimes (quantum) gravity effects may lead to potentially observable deformations of relativistic particles kinematics.
- If in the relevant regime a scale of mass is present, the deformation might exhibit itself through a nontrivial geometry of momentum space.
- As a result the spacetime symmetries and/or momentum composition laws might be deformed, with the deformation scale $1/\kappa$.

Kappa deformation

- Kappa-Poincare symmetry is a deformation of Poincare symmetry; it is a relativistic symmetry of systems with de Sitter momentum space (more precisely the momentum space is the AN(3) group manifold.)
- This is the only well-understood deformation of the Poincare group and it serves as a working model for general frameworks like DSR or Relative Locality.
- Kappa deformation can be constructed in any spacetime dimension.

Properties of kappa-deformation

- The deformation parameter κ is dimensionful and has the dimension of energy/mass/momentum;
- The only deformed commutators of kappa deformation belong to the boost-momentum sector

$$\left[N_i, k_j\right] = \delta_{ij}\left(\frac{\kappa}{2}\left(1 - e^{-2k_0/\kappa}\right) + \frac{\mathbf{k}^2}{2\kappa}\right) - \frac{1}{\kappa}k_i k_j$$

$$\left[N_i, k\_0\right] = k_i$$

Properties of kappa-deformation

- Spacetime is non-commutative; the spacetime non-commutativity has the structure of the AN(3) algebra

$$[x^0, x^i] = \frac{1}{\kappa} x^i$$

- Kappa-deformation possesses the structure of Hopf algebra (nontrivial coproduct and antipode $\leftrightarrow$ nontrivial multiparticle states.)

The challenge

- It is not excluded that kappa-Poincare is an *a priori* structure that replaces Poincare algebra at Planck scale, but here we assume that it is emergent and arises as a deformation resulting from (quantum) gravity.
- We do not know if this is the case in 3+1D; therefore we have to use toy models.
- Contrary to some claims, **it turns out that kappa deformation naturally arises in 2+1D, but with an interesting, unexpected twist.**

2+1D gravity and particles

- 2+1D gravity coupled to particles is „almost" topological: only the degrees of freedom at particle location are dynamical;
- These are the original DOF of particle and „would be" gauge DOF of gravity that become dynamical at the particle position.
- Combining the two we obtain the effective, deformed particle model.

Gravity in 2+1 D with particles

- The Lagrangian of 2+1 gravity with one (massive) particle at the origin is

$$L = \frac{1}{2\kappa}\int d^2x\, \epsilon^{ij}\left\langle \dot{A}_i A_j \right\rangle - \int d\tau \left\langle h^{-1}\dot{h}C \right\rangle$$
$$+\int d^2x \left\langle A_0 \left(\frac{k}{2\pi}\epsilon^{ij}F_{ij} - hCh^{-1}\delta^2(\vec{x}) \right) \right\rangle$$

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$C = mJ_0 + sP_0,$

h is gauge group element

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Constraint: curvature is nonzero only at the position of the particle

$h \in$ Poincare group

$C = mJ_0 + sP_0$

Gravity in 2+1 D with particles

$$L = \frac{1}{2\kappa}\int d^2x\, \epsilon^{ij}\left\langle \dot{A}_i A_j \right\rangle - \left\langle h^{-1}\dot{h}C \right\rangle$$

- The idea is to solve the constraint

$$\frac{1}{\kappa}\epsilon^{ij}F_{ij} = hCh^{-1}\delta^2(\vec{x})$$

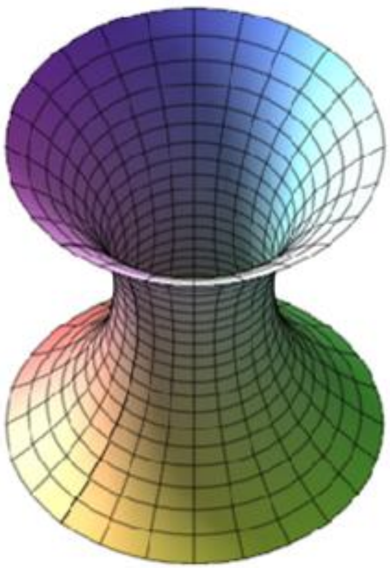
- and plug the solution back to the lagrangian. (This can be done explicitly – but only in the case of a simple enough gauge group)

A.Y. Alekseev and A.Z. Malkin, Commun. Math. Phys. 169, 99 (1995) [arXiv:hep-th/9312004; C. Meusburger and B. J. Schroers, Class. Quant. Grav. 20 (2003) 2193 [arXiv:gr-qc/0301108.

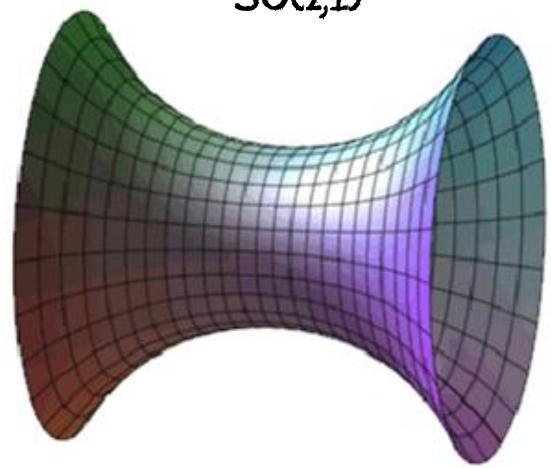
Gauge group of 2+1D gravity

- In the case of finite positive cosmological constant the gauge group is 2+1D de Sitter group SO(3,1) (Iwasawa decomposition).

AN(2)



SO(2,1)



- The mathematics is manageable only if one of the components is flattened after contraction

Standard Poincare contraction

$$[J_a, J_b] = \epsilon_{ab}^{\ \ c} J_c, \quad [J_a, P_b] = \epsilon_{ab}^{\ \ c} P_c, \quad [P_a, P_b] = -\epsilon_{ab}^{\ \ c} J_c$$

$$\downarrow$$

$$\bar{P} = \Lambda P, \quad \Lambda \to 0$$

$$\downarrow$$

$$[J_a, J_b] = \epsilon_{ab}^{\ \ c} J_c, \quad [J_a, \bar{P}_b] = \epsilon_{ab}^{\ \ c} \bar{P}_c, \quad [\bar{P}_a, \bar{P}_b] = 0$$

- This contraction flattens the AN(2) part and gives Poincare algebra;>the resulting deformed theory is well known (Lorentz double) and SO(2,1) (AdS) becomes the curved momentum space.

Another contraction

$$[J_a, J_b] = \epsilon_{ab}^{\ \ c} J_c, \ \ [J_a, P_b] = \epsilon_{ab}^{\ \ c} P_c, \ \ [P_a, P_b] = -\epsilon_{ab}^{\ \ c} J_c$$

$$S_a = P_a + \epsilon_{a0b} J^b$$

$$[J_a, J_b] = \epsilon_{ab}^{\ \ c} J_c, \ \ [J_a, S_b] = \epsilon_{ab}^{\ \ c} S_c - \eta_{ab} J_0 + \eta_{b0} J_a, \ \ [S_0, S_i] = S_i$$

$$\downarrow$$

$$\tilde{J}_a \equiv \sqrt{\Lambda} J_a, \ \ \Lambda \to 0$$

$$\downarrow$$

$$[\tilde{J}_a, \tilde{J}_b] = 0, \ \ [\tilde{J}_a, S_b] = (\eta_{b0} \tilde{J}_a - \eta_{ab} \tilde{J}_0), \ \ [S_0, S_i] = S_i$$

- This contraction flattens the SO(2,1) part and the resulting deformed theory has AN(2) (dS) as the curved momentum space.
- But the algebra is not the 2+1D kappa-Poincare!

Efective deformed particle

$$L = \frac{1}{2\kappa}\int d^2x\, \epsilon^{ij}\left\langle \dot{A}_i A_j \right\rangle - \left\langle h^{-1}\dot{h}C \right\rangle$$

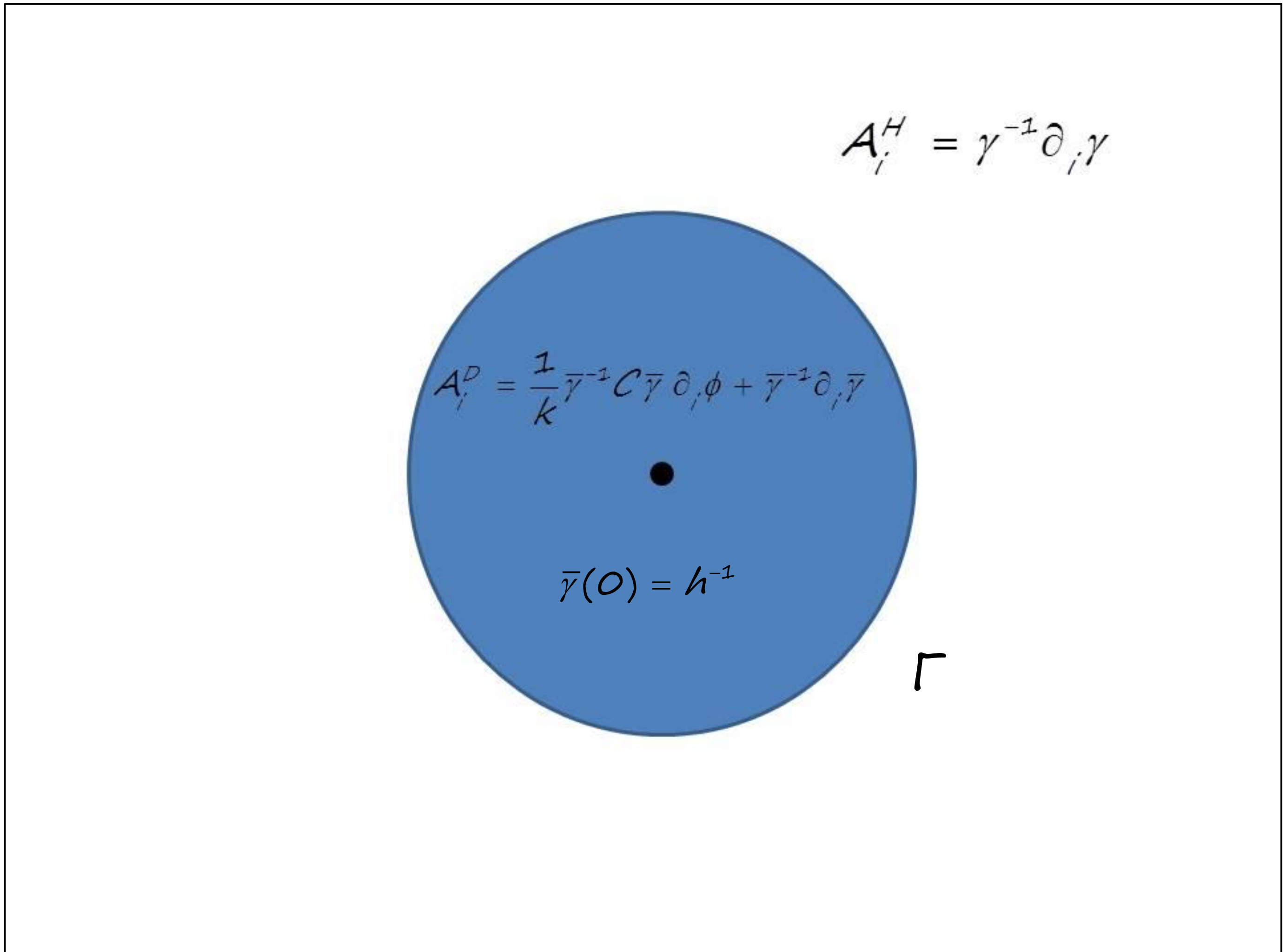
- with the constraint

$$\frac{1}{\kappa}\epsilon^{ij}F_{ij} = hCh^{-1}\delta^2(\vec{x})$$

- And the gauge algebra

$$[\tilde{J}_a, \tilde{J}_b] = 0, \quad [\tilde{J}_a, S_b] = (\eta_{b0}\tilde{J}_a - \eta_{ab}\tilde{J}_0), \quad [S_0, S_i] = S_i$$

Solving the constraint $\frac{1}{\kappa}\epsilon^{ij}F_{ij} = hCh^{-1}\delta^2(\vec{x})$



Continuity condition

$$A_i^H\Big|_\Gamma = A_i^D\Big|_\Gamma$$

Effective deformed particle

- At the end of the day the lagrangian of deformed particle is (in the spinless case)

$$L = -\kappa \left(\Pi \dot{\Pi}^{-1} \right)_a x^a$$

- where Π is an element of AN(2) of the form

$$\Pi \equiv \mathfrak{s}\, e^{\frac{m}{\kappa} S_0} \mathfrak{s}^{-1}$$

- Π is the equal to the holonomy of gravitational field (Lorentz sector) calculated along a loop surrounding the particle

Kappa-Carroll particle

- Π is the group valued momentum of the particle, but what are its properties? In general

$$\Pi = e^{p^i/\kappa S_i}\, e^{p^0/\kappa S_0}$$

- but if

$$\Pi \equiv \mathfrak{s}\, e^{\frac{m}{\kappa} S_0}\, \mathfrak{s}^{-1}$$

- Then the mass shell condition is

$$p_0 = m$$

- It is not the kappa-Poincare particle, but the kappa-Carroll one.

Kappa-Carroll particle

- In components the lagrangian has the form

$$L = x^0 \dot{p}_0 + x^i \dot{p}_i - \frac{1}{\kappa} x^i p_i \dot{p}_0 + \lambda(p_0^2 - m^2)$$

- Poisson brackets

$$\{x^0, x^i\} = \frac{1}{\kappa} x^i \ plus \ldots$$

Kappa-Carroll particle

- Equations of motion

$$\dot{x}^0 = 2\lambda p_0 = 2\lambda m, \quad \dot{x}^i = 0$$

- So indeed „it takes all the running you can do, to keep in the same place"

Comments

- There is room for kappa-physics in 2+1D gravity.
- However, the deformed particle does not have kappa-Poincare symmetry, but only kappa-Carroll one.
- It is unclear if it is possible to construct a particle model, derived from 2+1D gravity with full kappa-Poincare symmetry.

Comments: two particles

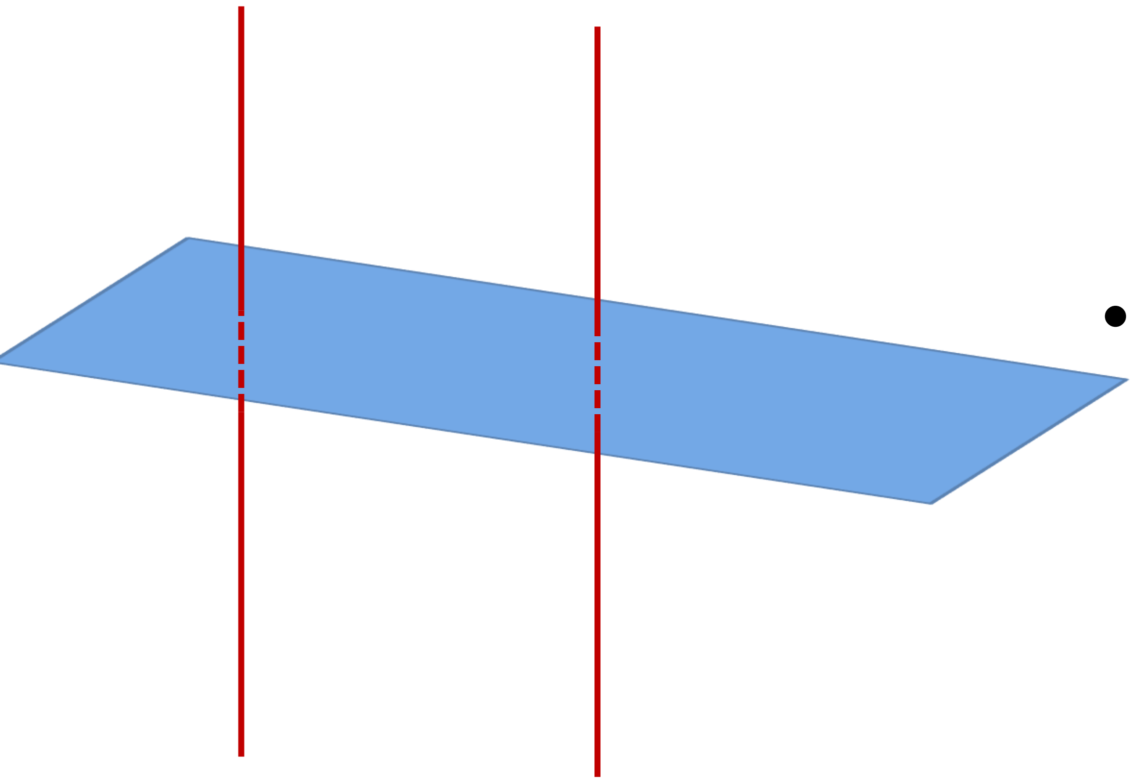
- It is possible to derive the lagrangian for two- and multi-particles systems. It reads (just kinetic term)

$$\mathcal{L} = \kappa\left(\dot{\Pi}_1 \Pi_1^{-1}\right)_a x_1^a + \kappa\left(\dot{\Pi}_2 \Pi_2^{-1}\right)_a x_2^a + \kappa\left(\Pi_2 \dot{\Pi}_1 \Pi_1^{-1} \Pi_2^{-1} - \dot{\Pi}_1 \Pi_1^{-1}\right)_a x_2^a$$

- Compared to the lagrangian of the two-particles system considered in Relative Locality framework this lagrangian has an additional „topological interaction term"
- The total conserved momentum is here

$$\Pi \equiv \Pi_2 \Pi_1$$

Comments: 2+1D vs RL



- In RL the total momentum is given by the undeformed sum $p=p_1+p_2$
- In 2+1D gravity construction the total momentum is given by the deformed sum $p=p_1 \oplus p_2$

Comments: 2+1D vs 3+1D

- Q: Is it possible to rewrite this lagrangian in 3+1D?
- A: Yes
- Q: Is it possible to repeat this construction in 3+1D?
- A: Don't know; certainly the essential element, the relation between group valued momenta and holonomies is lost. It is not clear what (if anything) could replace it.

Thank you