Non-Commutativity (and DSR) from Twist

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Talk given at Sapienza, CTCQG, Rome
September 10, 2014

NR-deformed QM based on:

- B. Chakraborty, Z. Kuznetsova, F.T., *Twist deformation of rotationally invariant quantum mechanics*, **J. Math. Phys.** 51 (2010) 112102.
- B. Chakraborty, P.G. Castro, R. Kullock and F.T., *Noncommutative oscillators from a Drinfeld twist deformation. A first principle derivation*, **J. Math. Phys.** 52 (2011) 032102.
- P.G. Castro, R. Kullock and F.T., *Snyder noncommutativity and pseudo-Hermitian Hamiltonians from a Jordanian twist*, **J. Math. Phys.** 52 (2011) 062105.
- F. T., *An Unfolded Quantization for Twisted Hopf Algebras*, **J.o.P. Conf. Ser.** 343 (2012) 012123.
- Z. Kuznetsova and F.T., *Effects of twisted noncommutativity in multi-particle Hamiltonians*, **Eur. Phys. J. C** 73 (2013) 2484.

Twist-deformed Poincaré (soon to appear):

jordanian versus extended jordanian twist/ (pseudo)-hermiticity/ non-additivity/ W-algebra, etc.

Different frameworks:

- kappa-Poincaré: Lukierski-Nowicki-Ruegg-Tolstoy.
- DSR: Amelino-Camelia, Kowalski-Glikman, Smolin, Freidel, ...
- Extended jordanian twist: Kulish et al.
- extended jordanian twist for Poincaré-Weyl and gl case: Borowiec-Pachol.
- extended jordanian twist and light-like kappa-Minkowski twist deformation of Poincaré: Meljanac et al.
- classification of Poincaré algebra deformations: not yet completed (Tolstoy-Zakrzewski partial results).

Drinfeld-twist-induced noncommutativity
(versus “fundamental” NC):
The NC-parameter is derived from the twist parameter.

Objective: First Quantization of non-relativistic QM, respecting the Hopf algebra structure.

Which is the “correct” Universal Enveloping Algebra?
(the one reproducing the physical inputs?)

Intensive versus extensive operators (including the additive subclass).

Additive operators: energy, momentum, ...
Non-additive operators: center-of-mass, ...

Hopf algebras:

Let A be an associative algebra over a field $F(= \mathbf{C})$ with an identity map $i : F \to A$ and a product $\mu : A \otimes A \to A$.

A is called a **Hopf algebra** if

a) there exists a homomorphism $\Delta : A \to A \otimes A$, called coproduct, satisfying the coassociativity condition $(id \otimes \Delta)(A) = (\Delta \otimes id)(A)$, and

b) there exists a homomorphism $\varepsilon : A \to F$ called counit and an antihomorphism $S : A \to A$ called antipode.

Diagrams relating coproduct, counit and antipode can be found, for example, in E. Abe, Hopf algebras, Cambridge tracts in mathematics 74, Cambridge Univ. Press, 1980.

The noncommutativity is obtained by deformation of the **universal enveloping algebra** $\mathcal{U}(\mathcal{G})$ with its Hopf algebraic structure.
The generators g_i of $\mathcal{G}$ are called the **primitive elements.**
For the primitive elements $g_i \in \mathcal{G}$ the undeformed costructures and the antipode are

$$
\begin{aligned}
\Delta(g_i) &= g_i \otimes \mathbf{1} + \mathbf{1} \otimes g_i \\
\varepsilon(g_i) &= 0 \\
S(g_i) &= -g_i
\end{aligned}
$$

For the identity $\mathbf{1} \in \mathcal{U}(\mathcal{G})$ the costructures and the antipode are

$$
\begin{aligned}
\Delta(\mathbf{1}) &= \mathbf{1} \otimes \mathbf{1} \\
\varepsilon(\mathbf{1}) &= 1 \\
S(\mathbf{1}) &= \mathbf{1}
\end{aligned}
$$

Physical interpretation of the coproduct:

additive operators $\rightarrow$ primitive generators of the Hopf algebra

$\Delta(g) = g \otimes \mathbf{1} + \mathbf{1} \otimes g$.

If g is associated to the undeformed Hamiltonian it implies the additivity the additivity of the two-particle energy levels:

$E_{1+2} = E_1 + E_2$.

Other applications of coproduct (and Hopf algebras): composition formulas

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) \quad \leftrightarrow \quad \Delta(\sin) = \sin \otimes \sin$$

$$\exp(x+y) = \exp(x)\exp(y) \quad \leftrightarrow \quad \Delta(\exp) = \exp \otimes \exp$$

Coproduct for the Universal Enveloping Heisenberg-Lie algebra:

$\mathcal{H} = \{a, a^\dagger, \hbar\}$

$$\begin{aligned}
\Delta(a^\dagger a) &= \Delta(a^\dagger)\Delta(a) = \\
&= (a^\dagger \otimes \mathbf{1} + \mathbf{1} \otimes a^\dagger)(a \otimes \mathbf{1} + \mathbf{1} \otimes a) = \\
&= (a^\dagger a) \otimes \mathbf{1} + \mathbf{1} \otimes (a^\dagger a) + a^\dagger \otimes a + a \otimes a^\dagger \neq \\
&\neq (a^\dagger a) \otimes \mathbf{1} + \mathbf{1} \otimes (a^\dagger a)
\end{aligned}$$

Framework: "Unfolded Quantization"

based on an unfolded Lie algebra $\mathcal{G}$ such that

$\mathcal{G} \supset$ the Heisenberg-Lie algebra as a subalgebra.
$\mathcal{G} \supset$ the dynamical Lie algebra as a subalgebra.

A $2D$ example:

$$\mathcal{G} = \{\hbar, x_1, x_2, p_1, p_2, X_{11}, X_{22}, X_S, P_{11}, P_{22}, P_S, M_{11}, M_{22}, M_{12}, M_{21}\}.$$

Heisenberg-Lie subalgebra:

$$[x_i, p_j] = i\hbar\delta_{ij}.$$

Explicit realization of the $\mathcal{G}$-Lie brackets:

$$\begin{array}{lll}
\overline{x}_i = x_i, & \overline{p}_i = p_i, & \overline{\hbar} = \hbar, \\
\overline{X}_{11} = \frac{1}{\hbar}{x_1}^2, & \overline{X}_{22} = \frac{1}{\hbar}{x_2}^2, & \overline{X}_S = \frac{1}{\hbar}(x_1x_2 + x_2x_1), \\
\overline{P}_{11} = \frac{1}{\hbar}{p_1}^2, & \overline{P}_{22} = \frac{1}{\hbar}{p_2}^2, & \overline{P}_S = \frac{1}{\hbar}(p_1p_2 + p_2p_1), \\
\overline{M}_{11} = \frac{1}{\hbar}(x_1p_1 + p_1x_1), & \overline{M}_{22} = \frac{1}{\hbar}(x_2p_2 + p_2x_2), & \\
\overline{M}_{12} = \frac{1}{\hbar}(x_1p_2 + p_2x_1), & \overline{M}_{21} = \frac{1}{\hbar}(x_2p_1 + p_1x_2). &
\end{array}$$

An Unfolded Lie algebra $\mathcal{G}$:

$$
\begin{gathered}
[x_i, p_j] = i\hbar\delta_{ij}, \\
[x_i, P_{jj}] = 2i\delta_{ij}p_j, \quad [p_i, X_{jj}] = -2\delta_{ij}x_j, \\
[x_i, M_{jk}] = 2i\delta_{ik}x_j, \quad [p_i, M_{jk}] = -2i\delta_{ij}p_k, \\
[x_1, P_S] = 2ip_2, \quad [x_2, P_S] = 2ip_1, \quad [p_1, X_S] = -2ix_2, \quad [p_2, X_S] = -2ix_1, \\
[X_{ii}, P_{jj}] = 2i\delta_{ij}M_{ij}, \\
[X_{11}, P_S] = [X_S, P_{22}] = 2iM_{12}, \quad [X_{22}, P_S] = [X_S, P_{11}] = 2iM_{21}, \\
[X_S, P_S] = 2i(M_{11} + M_{22}), \\
[X_{ii}, M_{ii}] = 4iX_{ii}, \quad [X_{11}, M_{21}] = [X_{22}, M_{12}] = 2iX_S, \\
[P_{ii}, M_{ii}] = -4iP_{ii}, \quad [P_{11}, M_{12}] = [P_{22}, M_{21}] = -2iP_S, \\
[X_S, M_{ii}] = 2iX_S, \quad [X_S, M_{12}] = 4iX_{11}, \quad [X_S, M_{21}] = 4iX_{22}, \\
[P_S, M_{ii}] = -2iP_S, \quad [P_S, M_{12}] = -4iP_{22}, \quad [P_S, M_{21}] = -4iP_{11}, \\
[M_{11}, M_{12}] = -[M_{22}, M_{12}] = -2iM_{12}, \quad [M_{11}, M_{21}] = -[M_{22}, M_{21}] = 2iM_{21}, \\
[M_{12}, M_{21}] = 2i(M_{22} - M_{11}).
\end{gathered}
$$

The remaining commutators are vanishing.

Hopf algebra structure $\mathcal{U}(\mathcal{G})$.

Let V be a Hilbert space.

Mapping

$$\rho \quad : \quad \mathcal{U}(\mathcal{G}) \rightarrow End(V).$$

Then

$$\Omega \in \mathcal{U}(\mathcal{G}) \quad \mapsto \quad \widehat{\Omega} = \rho(\Omega) \in End(V).$$

We have:

$\widehat{\Omega^{(1)}} = \widehat{\Omega}$, 1-particle operator,
$\widehat{\Omega^{(2)}} = \widehat{\Delta(\Omega)}$, 2-particle operator,
$\widehat{\Omega^{(3)}} = (\Delta \widehat{\oplus \mathbf{1})}(\Omega) = (\mathbf{1} \widehat{\oplus \Delta)}(\Omega)$, 3-particle operator,
$\ldots$.

Physical interpretation of $\widehat{\hbar^{(n)}}$:

$\widehat{\hbar^{(n)}} = n$.
It is the particle-number operator.

Covariant formulas:

For 2-particle, e.g. $[x_{i,CM}, p_{j,tot}] = i\delta_{ij}\hbar$
is reexpressed through $[\Delta(x_i), \Delta(p_j)] = i\delta_{ij}\Delta(\hbar)$.

A class of primitive elements $\Omega \in \mathcal{G}$:

$$\Omega = a(P_{11} + P_{22}) + b(X_{11} + X_{22}) + c(M_{12} - M_{21}) + dx_1 + fp_2,$$

for a, b, c, d, f arbitrary real parameters.
The $\widehat{\Omega}$ operators are Hermitian.

Three special cases:

i) for $b = 1$ and $a = c = d = f = 0$, Ω coincides with the "squared radius" $R^2 = X_{11} + X_{22}$;

ii) for $a = \frac{1}{2}$ and $b = \frac{1}{2}\omega^2$ Ω is associated to the Hamiltonian of the harmonic oscillator $H = \frac{1}{2}(P_{11} + P_{22}) + \frac{1}{2}\omega^2(X_{11} + X_{22})$;

iii) for $a = \frac{1}{2m}$, $b = \frac{m{\omega_c}^2}{8}$, $c = \frac{\omega_c}{4}$, $d = eE$, $f = 0$, we obtain the Quantum Hall Effect Hamiltonian in the presence of constant electric (E) and magnetic (B) fields (e is the electron's charge, $\omega_c = \frac{eB}{mc}$ is the cyclotron frequence).

Drinfel'd twist

The **Drinfel'd twist** $\mathcal{F} \in \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})$ should satisfy cocycle and counitarity conditions

1) $(\mathbf{1} \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F} = (\mathcal{F} \otimes \mathbf{1})(\Delta \otimes id)\mathcal{F}$

2) $(\varepsilon \otimes id)\mathcal{F} = \mathbf{1} = (id \otimes \varepsilon)\mathcal{F}$

The **deformed co-structures**, for $a \in \mathcal{U}(\mathcal{G})$, are:

$\Delta^{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}$

$S^{\mathcal{F}}(a) = \chi S(a)\chi^{-1}$

with $\chi = \mu(id \otimes S)\mathcal{F}$.

In order for find linear subspace of $\mathcal{U}(\mathcal{G})$ one should calculate the **deformed generators** $g_i \mapsto g_i^{\mathcal{F}} = \bar{f}^{\alpha}(g_i)\bar{f}_{\alpha}$.

Here the Sweedler notation $\mathcal{F}^{-1} \equiv \bar{f}^{\alpha} \otimes \bar{f}_{\alpha}$ has been used.

Twist deformation (abelian case), induced by $\mathcal{F} \in \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})$:

$$\mathcal{F}^{-1} = e^{-i\alpha\epsilon_{ij}p_i \otimes p_j} \equiv \bar{f}^{\beta} \otimes \bar{f}_{\beta}$$

Deformed generators $\tau^{\mathcal{F}}$ induced by the twist:

$$\tau \in \mathcal{G} \quad \mapsto \tau^{\mathcal{F}} = \bar{f}^{\beta}(\tau)\bar{f}_{\beta} \in \mathcal{U}(\mathcal{G}).$$

Bopp shift recovered:

$$\hbar^{\mathcal{F}} = \hbar, \quad {p_i}^{\mathcal{F}} = p_i, \quad {x_i}^{\mathcal{F}} = x_i - \alpha\epsilon_{ij}\hbar p_j,$$

we have that

$$[{x_1}^{\mathcal{F}}, {x_2}^{\mathcal{F}}] = \Theta = 2i\alpha\hbar^2$$

Comment: the NC parameter Θ is derived.

The twist maps $\Omega \in \mathcal{G}$ into $\Omega^{\mathcal{F}} \in \mathcal{U}(\mathcal{G})$:

$$\Omega^{\mathcal{F}} = \Omega + \alpha[2b(x_1p_2 - p_2x_1) - 2c({p_1}^2 + {p_2}^2) - d\hbar p_2] + \alpha^2 b\hbar({p_1}^2 + {p_2}^2).$$

Single-particle Quantization

$$\Omega^{\mathcal{F}} \in \mathcal{U}(\mathcal{G}) \to \widehat{\Omega^{\mathcal{F}}} \in End(V):$$

.

$$\widehat{\Omega}^{\mathcal{F}} = s(N+1) + tZ \qquad (s \geq |t|),$$

in terms of the commuting operators N, Z ($[N, Z] = 0$)

$$N = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2, \quad Z = i(\hat{a}_2 \hat{a}_1^\dagger - \hat{a}_1 \hat{a}_2^\dagger).$$

Eigenvalues: $n = 0, 1, 2, \ldots$,
$z = -n + 2j$ ($j = 0, 1, \ldots, n$).

For $s = |t|$, the vacuum is infinitely degenerate.
A unique vacuum solution exists for $s > |t|$.

Creation and annihilation operators

$$a_i{}^{(\lambda)} := \frac{1}{\sqrt{2}}(\lambda x_i + i\frac{p_i}{\lambda}) \quad , \quad a_i{}^{(\lambda)\dagger} := \frac{1}{\sqrt{2}}(\lambda x_i - i\frac{p_i}{\lambda}),$$

with λ suitably chosen.

The three cases:

i) the deformed squared radius operator $\widehat{R^2}^{\mathcal{F}}$

$$\lambda = \frac{1}{\sqrt{\alpha}} \quad , \quad s = t = 2\alpha$$

(one should note the singular limit for $\alpha \to 0$);

ii) the deformed hamiltonian $\widehat{H^{\mathcal{F}}}$ of the harmonic oscillator

$$\lambda = \sqrt[4]{\frac{\omega^2}{1+\alpha^2\omega^2}} \quad , \quad s = \omega\sqrt{1+\alpha^2\omega^2}, \qquad t = \alpha\omega^2;$$

iii) the deformed hamiltonian $\widehat{H_{QHE}}^{\mathcal{F}}$, in the presence of a constant magnetic field B

$$\lambda = \sqrt[4]{\frac{m\omega_c}{2 - m\alpha\omega_c}} \quad , \quad s = -t = \frac{1}{2}\omega_c(1 - \frac{\alpha\omega_c}{4}).$$

Different NC-quantizations for single-particle case in the literature:

i) the deformed squared radius operator
Scholtz, Gouba, Hafver, Rohwer, J. Phys. A (2009)
(quantization of the configuration space).

ii) the deformed hamiltonian of the harmonic oscillator
Kijanka, Kosinski, Phys. Rev. D (2004)

iii) the NC-quantum Hall Effect Hamiltonian
Dayi, Jellal, J. Math. Phys. (2002).

For the single-particle spectrum the “Unfolded Quantization” recovers their results.

The novelty: the multi-particle sector

Coproduct or twisted coproduct? They are unitarily equivalent:

$$\forall \tau \in \mathcal{U}(\mathcal{G}), \quad \Delta^{\mathcal{F}}(\tau) = \mathcal{F}\Delta(\tau)\mathcal{F}^{-1} \in \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}).$$

Applied on $V \otimes V$ the twist becomes a unitary operator $F \in End(V \otimes V)$:

$$\widehat{\Delta^{\mathcal{F}}(\tau)} = F\widehat{\Delta(\tau)}F^{-1} \in End(V \otimes V).$$

$\Delta(\Omega^{\mathcal{F}}) \in \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G})$ is

$$\begin{aligned}\Delta(\Omega^{\mathcal{F}}) &= \Delta(\Omega) + 2\alpha b\Delta(x_2 p_1 - x_1 p_2) - 2\alpha c\Delta(p_1^2 + p_2^2) \\ &\quad -\alpha d\Delta(\hbar p_2) + \alpha^2 b\Delta(\hbar(p_1^2 + p_2^2)).\end{aligned}$$

Therefore $\widehat{\Delta(\Omega^{\mathcal{F}})} \in End(V \otimes V)$ is

$$\widehat{\Delta(\Omega^{\mathcal{F}})} = \widehat{\Omega^{\mathcal{F}}} \otimes \mathbf{1} + \mathbf{1} \otimes \widehat{\Omega^{\mathcal{F}}} + \widehat{\Omega}_r \otimes \mathbf{1} + \mathbf{1} \otimes \widehat{\Omega}_r + \widehat{\Omega}_{mix},$$

where

$$\widehat{\Omega}_r = -\alpha d\hat{p}_2 + \alpha^2 b(\hat{p}_1^2 + \hat{p}_2^2) \in End(V)$$

and

$$\begin{aligned} \widehat{\Omega}_{mix} = {} & 2\alpha b[\hat{x}_2 \otimes \hat{p}_1 + \hat{p}_1 \otimes \hat{x}_2 - \hat{x}_1 \otimes \hat{p}_2 - \hat{p}_2 \otimes \hat{x}_1] \\ & -4\alpha(c - \alpha b)[\hat{p}_1 \otimes \hat{p}_1 + \hat{p}_2 \otimes \hat{p}_2] \in End(V \otimes V). \end{aligned}$$

The non-additivity is implied by the presence of the last three terms in the right hand side.

Application: harmonic oscillator

Three cases:
a) the undeformed 2-particle Hamiltonian,
b) the “additive” NC 2-particle Hamiltonian constructed as a sum of two 1-particle NC Hamiltonians,
c) the twist-deformed 2-particle Hamiltonian.

The spectrum

(for n_1, n_2, j_1, j_2 with $-n_1 \leq j_1 \leq n_1$, $-n_2 \leq j_2 \leq n_2$):

$$a): \quad E_{n_1,n_2,j_1,j_2} = 2\omega(n_1+n_2)+4\omega.$$

$$b): \quad E_{n_1,n_2,j_1,j_2} = 2\omega\sqrt{1+\alpha^2\omega^2}(n_1+n_2+2)+2\alpha\omega^2(j_1+j_2).$$

$$c): \quad E_{n_1,n_2,j_1,j_2} = 2\omega\sqrt{1+4\alpha^2\omega^2}(n_1+1)+2\omega(n_2+1)+4\alpha\omega^2 j_1.$$

The n-particle spectrum for $n \geq 3$:

The coassociativity of the coproduct

$$(id \otimes \Delta)\Delta(\Omega^{\mathcal{F}}) = (\Delta \otimes id)\Delta(\Omega^{\mathcal{F}}) \equiv \Delta_{(2)}(\Omega^{\mathcal{F}})$$

implies the associativity of the spectrum

$$\begin{aligned} E^{\mathcal{F}}_{123} &\equiv E^{\mathcal{F}}_{(12)3} = E^{\mathcal{F}}_{1(23)} = \\ &= E^{\mathcal{F}}_1 + E^{\mathcal{F}}_2 + E^{\mathcal{F}}_3 + \Omega_{12} + \Omega_{23} + \Omega_{31} + \Omega_{123}, \end{aligned}$$

with Ω_{123} recovered from the Ω_{ij}'s.

NC breaking of rotational invariance

(Chakraborty,Kuznetsova, F.T. in JMP).

The original $su(2)$ rotational algebra is recovered in terms of the $\mathcal{F}$-commutator of the twisted angular momentum:

$$[L_i{}^{\mathcal{F}}, L_j{}^{\mathcal{F}}]_{\mathcal{F}} = i\epsilon_{ijk} L_k{}^{\mathcal{F}}.$$

The rotational symmetry is preserved by the twist-deformation and, since,

$$\begin{aligned} [x_i^{\mathcal{F}}, L_j^{\mathcal{F}}]_{\mathcal{F}} &= i\epsilon_{ijk} x_k^{\mathcal{F}}, \\ [p_i^{\mathcal{F}}, L_j^{\mathcal{F}}]_{\mathcal{F}} &= [p_i, L_j^{\mathcal{F}}]_{\mathcal{F}} = i\epsilon_{ijk} p_k^{\mathcal{F}}, \end{aligned}$$

both $x_i^{\mathcal{F}}$ and p_i have vectorial transformation properties under the deformed brackets.

On the other hand. What happens to operators which are rotationally invariant in the undeformed case? Do they keep the rotational invariant property even in the deformed case or otherwise acquire an anomalous term which disappears in the limit $\vec{\rho} \to 0$?

The answer can be given by checking the commutation relations

$$[{L_i}^{\mathcal{F}}, B^\sharp]_{\mathcal{F}}$$

for an operator $B^\sharp$ belonging to the Universal Enveloping Algebra of a Lie algebra containing the Euclidean algebra $e(3)$ as a subalgebra and such that $B^\sharp$ is expanded in $\vec{\rho}$ Taylor series:

$$B^\sharp \;=\; B_0 + B_1 + B_2 + \ldots,$$

with B_k k-linear in $\vec{\rho}$. Here $B_0 \equiv B$ denotes the undeformed limit for $\vec{\rho} \to 0$ of $B^\sharp$ (we can therefore say that the operator $B^\sharp$ is the deformation of B).
The rotational invariance in the undeformed limit requires that the following relation involving ordinary commutators and angular momentum operators has to be satisfied

$$[L_i, B_0] \;=\; 0.$$

We get the recursion relations:

$$[L_i{}^{\mathcal{F}}, B^\sharp]_{\mathcal{F}} \quad = \quad [L_i - K_i, B^\sharp] + M_{ik}[p_k, B^\sharp],$$

with M_{ik} given by

$$M_{ik} \quad = \quad 2\rho_k p_i - 2\rho_i p_k.$$

The result is that a rotationally invariant operator B such that

$$[L_i, B] \quad = \quad 0.$$

can develop, under deformation, an anomaly A_i which is expressed through

$$[L_i{}^{\mathcal{F}}, B^\sharp]_{\mathcal{F}} \quad = \quad A_i.$$

Example (Coulomb potential):

$$\left(\frac{1}{r}\right)^\sharp \quad = \quad \frac{1}{r} - \hbar\epsilon_{ljk}\rho_l p_j x_k \frac{1}{r^3} + O(\hbar^2),$$

satisfies the anomalous twist-deformed commutator with minimal anomaly:

$$\left[L_i{}^{\mathcal{F}}, \left(\frac{1}{r}\right)^\sharp\right] \quad = \quad \hbar^2\left(\frac{\rho_i}{r^3} - 3x_i\frac{\vec{\rho}\vec{x}}{r^5}\right) + O(\hbar^3).$$

More general twists within the Unfolded Quantization:

$\mathcal{U}(sl(2))$ Jordanian twist $\Rightarrow$ Snyder NC.

Unfolded Lie algebra $\mathcal{G} = \{\hbar, x_i, p_i, H, K, D\}$,
(Heisenberg-Lie and $sl(2)$ are subalgebras).

Jordanian twist: $\mathcal{F} = \exp\left(-iD \otimes \sigma\right)$,
where $\sigma = \ln(\mathbf{1} + \xi H)$, with α real.

Then $x_i^{\mathcal{F}} = x_i e^{\frac{\sigma}{2}}, p_i^{\mathcal{F}} = p_i e^{-\frac{\sigma}{2}}$.

The Snyder noncommutativity recovered through

$$[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}] \quad = \quad -\frac{i\xi}{2}(x_i^{\mathcal{F}} p_j^{\mathcal{F}} - x_j^{\mathcal{F}} p_i^{\mathcal{F}}).$$

Jordanian Twist

There are only two inequivalent deformations of $sl(2)$. The first one (P. P. Kulish and N. Yu. Reshetikhin, J. Sov. Math. **23** (1983) 2435; M. Jimbo, Lett. Math. Phys. **10** (1985) 63), depends on a non-dimensional parameter q; it leads to the quantum group $\mathcal{U}_q(sl(2))$ and cannot be obtained from Drinfel'd twist technique.

The second one is called the Jordanian deformation of $sl(2)$. It can be obtained from the twist

$$\mathcal{F} = \exp\left(-iD \otimes \sigma\right)$$

where $\sigma = \ln(\mathbf{1} + \xi H)$, and H, D, K are $sl(2)$ generators. Dubois-Violette and Launer (1990), Ohn (1992), Ogievetsky (1993), Kulish and Celeghini (1998), Boroviec, Lukierski and Tolstoy (2003).

We can consider different differential realizations of $sl(2)$.

- 1st order differential realization:

$$\begin{aligned} H &= i\partial_t, \\ D &= -it\partial_t + \beta, \\ K &= it^2\partial_t - 2\beta t; \end{aligned}$$

  from the hermiticity condition we have $\beta = -i/2 + \lambda$, $\lambda \in \mathbf{R}$.

- 2nd order differential realization:

$$\begin{aligned} H &= -\partial_x^2 + \frac{\rho}{x^2}, \\ D &= -\frac{i}{2}x\partial_x + c, \\ K &= \frac{1}{4}x^2, \end{aligned}$$

  for arbitrary ρ and $c = i/4$.
  The second one can act on d-dimensional space $(x_1, \ldots, x_d)$.

Two different cases of Jordanian twist with 2nd order differential realizations of $sl(2)$:

1. $\rho = 0$ - free particle or harmonic oscillator Hamiltonian in non-deformed case.
   Unfolded algebra $\mathcal{G} = \{\hbar, x_i, p_j, H, D, K\}$.
2. $\rho \neq 0$ - "Calogero type" Hamiltonian.
   Unfolded algebra $\mathcal{G}$ is infinite-dimensional
   $$\mathcal{G} = \{\hbar, x_i, p_j, H, D, K, \frac{1}{r^4}, \frac{x_i}{r^4}, \frac{1}{r^4}p_i + p_i\frac{1}{r^4}, \frac{x_ix_j + x_jx_i}{r^6}, \frac{x_ix_j + x_jx_i}{r^6}p_j + p_j\frac{x_ix_j + x_jx_i}{r^6}, \ldots\}$$
   All combinations written above should be multiplied by certain powers of $\hbar$ and considered as primitive elements of the algebra.

The unfolded algebra for the case 2 in one-dimensional case.

This unfolded algebra is a subalgebra of the algebra of integer potentials of two types of generators (plus central element) which are either $[x, p] = i\hbar$ or $[a, a^\dagger] = \hbar$.

Denoting as a, b the operators and c the central element we have $\{a, b\} = c$ for Poisson brackets and $[a, b] = c$ for commutators.

For Poisson brackets we have
$\{b^n a^p, b^m a^q\} = (mp - nq)b^{n+m-1} a^{p+q-1} c$
or with $W_{n,p} \equiv \dfrac{b^n a^p}{c^{n+p-1}}$
it can be written as $\{W_{n,p}, W_{m,q}\} = (mp - nq)W_{n+m-1,p+q-1}$.

If the second index is $= 1$, defining $U_n \equiv W_{n+1,1}$ we see that it reproduces the centerless Virasoro algebra
$\{U_n, U_m\} = (m - n)U_{m+n}$.

recursive relations for commutators

$$
\begin{aligned}
[b^n a^p, b^m a^q] &= b^n [a^p, b^m] a^q + b^m [b^n, a^q] a^p \\
[a^p, b^n] &= \sum_{j=1}^{n} \frac{p!}{(p-j)!} \binom{n}{j} c^j b^{n-j} a^{p-j} \equiv c_{p,n}
\end{aligned}
$$

If $c = \hbar$, we have higher powers of $\hbar$.

The Jordanian twist of $\mathcal{U}(\mathcal{G})$

The twist induces a deformation $g \mapsto g^{\mathcal{F}}$. The deformed generators are the same for both unfolded algebras $\mathcal{G}$:

$$\begin{aligned} x_i^{\mathcal{F}} &= x_i e^{\frac{\sigma}{2}}, \quad p_i^{\mathcal{F}} = p_i e^{-\frac{\sigma}{2}}, \quad \hbar^{\mathcal{F}} = \hbar \\ H^{\mathcal{F}} &= H e^{-\sigma}, \quad K^{\mathcal{F}} = K e^{\sigma}, \quad D^{\mathcal{F}} = D. \end{aligned}$$

The commutator of the deformed position variables has the form:

$$[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}] = -\frac{i\xi}{2}\left(x_i^{\mathcal{F}} p_j^{\mathcal{F}} - x_j^{\mathcal{F}} p_i^{\mathcal{F}}\right) + \rho\mathcal{O}(\xi^3).$$

For $\rho = 0$ (or up to third order in ξ) we have the noncommutativity introduced by Snyder in H. S. Snyder, Phys. Rev. **71** (1947) 38.

Twisted commutators

We present the ordinary nonvanishing commutators of the deformed generators for the case 1:

$$[x_i^{\mathcal{F}}, p_j^{\mathcal{F}}] = i\hbar\delta_{ij} - (i\xi/2)p_i^{\mathcal{F}} p_j^{\mathcal{F}}$$
$$[x_i^{\mathcal{F}}, x_j^{\mathcal{F}}] = (i\xi/2)\left(x_j^{\mathcal{F}} p_i^{\mathcal{F}} - x_i^{\mathcal{F}} p_j^{\mathcal{F}}\right)$$

$$[x_i^{\mathcal{F}}, H^{\mathcal{F}}] = ip_i^{\mathcal{F}}\left(1 - \xi H^{\mathcal{F}}\right)$$
$$[x_i^{\mathcal{F}}, K^{\mathcal{F}}] = i\xi\left(K^{\mathcal{F}} p_i^{\mathcal{F}} - x_i^{\mathcal{F}} D^{\mathcal{F}}\right) + (3/4)\xi^2 x_i^{\mathcal{F}} H^{\mathcal{F}}$$
$$[x_i^{\mathcal{F}}, D^{\mathcal{F}}] = (i/2)x_i^{\mathcal{F}}\left(1 - \xi H^{\mathcal{F}}\right)$$
$$[p_i^{\mathcal{F}}, K^{\mathcal{F}}] = -i\left(x_i^{\mathcal{F}} + \xi p_i^{\mathcal{F}} D^{\mathcal{F}}\right) + (\xi^2/4)p_i^{\mathcal{F}} H^{\mathcal{F}}$$
$$[p_i^{\mathcal{F}}, D^{\mathcal{F}}] = -ip_i^{\mathcal{F}}\left(1 - (\xi/2)H^{\mathcal{F}}\right)$$

$$[D^{\mathcal{F}}, H^{\mathcal{F}}] = iH^{\mathcal{F}}\left(1 - \xi H^{\mathcal{F}}\right)$$
$$[D^{\mathcal{F}}, K^{\mathcal{F}}] = -iK^{\mathcal{F}}\left(1 - \xi H^{\mathcal{F}}\right)$$
$$[K^{\mathcal{F}}, H^{\mathcal{F}}] = 2iD^{\mathcal{F}}\left(1 + \xi H^{\mathcal{F}}\right) + 2\xi H^{\mathcal{F}} - 2\xi^2\left(H^{\mathcal{F}}\right)^2$$

Pseudo-hermitian Hamiltonians.

For the harmonic oscillator the undeformed Hamiltonian is
$\mathbf{H} = H + K$.
The deformed Hamiltonian

$$\mathbf{H}^{\mathcal{F}} = H^{\mathcal{F}} + K^{\mathcal{F}} = He^{-\sigma} + Ke^{\sigma}$$

is η-pseudo-Hermitian:

$$\mathbf{H}^{\mathcal{F}\dagger} = \eta \mathbf{H}^{\mathcal{F}} \eta^{-1},$$

where $\eta = e^{\sigma} = \mathbf{1} + \xi H$.
Mostafazadeh construction: the Hamiltonian becomes self-adjoint under the η-deformed inner product

$$<< \psi, \phi >> = < \psi, \eta\phi > .$$

As a vector space the Hilbert space $\tilde{\mathcal{H}}$ endowed with the η-deformed inner product is isomorphic to the original one, $\mathcal{H}$, as a Hilbert space this is not true. If η is positive definite, the new inner product will also be so.

All observables on $\tilde{\mathcal{H}}$ can be mapped back onto $\mathcal{H}$ where the inner product is the usual one, through the non-local transformation

$$\mathbf{H}^{\mathcal{F}} \mapsto \mathbf{H}^{\mathcal{F}}_{\rho} = \rho \mathbf{H}^{\mathcal{F}} \rho^{-1}.$$

with $\rho = \exp \frac{1}{2}\sigma$.
The new Hamiltonian, given by

$$\mathbf{H}^{\mathcal{F}}_{\rho} = \left(1 - \frac{\xi^2}{4}\right) H^{\mathcal{F}} + K^{\mathcal{F}} + i\xi D,$$

is explicitly Hermitian since $K^{\mathcal{F}\dagger} = K^{\mathcal{F}} + 2i\xi D$.
The transformation ρ is called a pseudo-canonical transformation. The systems described by $\mathbf{H}^{\mathcal{F}}$ in $\tilde{\mathcal{H}}$ and $\mathbf{H}^{\mathcal{F}}_{\rho}$ in $\mathcal{H}$ are physically equivalent.

The twist-deformed hermitian Hamiltonian, acting on V, is

$$\widehat{\mathbf{H}_\rho} = \left(1 - \frac{\xi^2}{4}\right) \frac{p_i p_i}{2 + \xi p_i p_i} + \frac{x_i x_i}{2} \left(1 + \xi \frac{p_i p_i}{2}\right) + i\xi \frac{x_i p_i + p_i x_i}{4}.$$

Most general Hamiltonian operator

$$
\begin{aligned}
\tilde{H} &= H_\rho + (\bar{\rho} - \rho)K^{-1} + \lambda K, \\
\tilde{H}^{\mathcal{F}} &= H_\rho^{\mathcal{F}} + (\bar{\rho} - \rho)(K^{-1})^{\mathcal{F}} + K^{\mathcal{F}} \\
&= H_\rho T^{-2} + (\bar{\rho} - \rho)K^{-1}T^{-2} + \lambda K T^2
\end{aligned}
$$

where $T = e^{\frac{\sigma}{2}}$.

Two types of η-hermiticity:

1. $\bar{\rho} - \rho \neq 0,\ \lambda = 0$
   $\left[H_\rho^{\mathcal{F}} + (\bar{\rho} - \rho)(K^{-1})^{\mathcal{F}}\right]^\dagger = \eta \left(H_\rho^{\mathcal{F}} + (\bar{\rho} - \rho)(K^{-1})^{\mathcal{F}}\right)\eta^{-1}$
   with $\eta = T^{-2}$,
2. $\bar{\rho} - \rho = 0,\ \lambda \neq 0$
   $\left[H_\rho^{\mathcal{F}} + \lambda K^{\mathcal{F}}\right]^\dagger = \eta \left(H_\rho^{\mathcal{F}} + \lambda K^{\mathcal{F}}\right)\eta^{-1}$ with $\eta = T^2$

II Part: twist and DSR

Two inequivalent two-dimensional Lie algebras over $\mathbb{C}$:
i) the abelian algebra $[a, b] = 0$ and
ii) the non-abelian algebra $[a, b] = ib$.

The non-abelian algebra *ii* is the Borel subalgebra of $sl(2)$:

$$\begin{aligned} [D, H] &= iH, \\ [D, K] &= -iK, \\ [K, H] &= 2iD. \end{aligned}$$

We can identify $D \equiv a$ and $H \equiv b$.

the *ii* algebra is a subalgebra of the Poincaré algebras $\mathcal{P}(d)$ and of the centrally extended $2D$ Poincaré algebra $\mathcal{P}_C(2)$:

$$\begin{aligned} [P_0, P_1] &= iC, \\ [M_{01}, P_0] &= -iP_1, \\ [M_{01}, P_1] &= iP_0. \end{aligned}$$

The $2D$ Poincaré algebra $\mathcal{P}(2)$ is recovered by letting $C = 0$.
The identifications are $a \equiv -M_{01}$, $b \equiv P_0 + P_1$.

The abelian algebra *i* induces the abelian twist

$$\mathcal{F} = \exp(-i\rho(a \otimes b - b \otimes a)),$$

where ρ is the (dimensional) deformation parameter.

The non-abelian algebra *ii* induces the (jordanian) twist

$$\mathcal{F} = \exp(-ia \otimes \ln(1 + \rho b)),$$

where ρ is the (dimensional) deformation parameter.

Under the transposition operator $\tau(v \otimes w) = w \otimes v$, the transposed twist

$$\mathcal{F}_\tau := \exp(-i \ln(1 + \rho b) \otimes a),$$

still satisfies the cocycle condition.

The jordanian and extended jordanian twist of the d-dimensional Poincaré algebra are

$$\mathcal{F} = \exp\left(iM \otimes \ln(1+\rho P_+) + i\epsilon M_{+j} \otimes \ln(1+\rho P_+)\frac{P_j}{P_+}\right).$$

The jordanian case is recovered for $\epsilon = 0$;
the extended jordanian case is recovered for $\epsilon = 1$.

Under transposition, the $\mathcal{F}_\tau$ twists are

$$\mathcal{F}_\tau = \exp\left(i\ln(1+\rho P_+) \otimes M + i\epsilon \ln(1+\rho P_+)\frac{P_j}{P_+} \otimes M_{+j}\right).$$

Under twist-deformation, a generator $g \in \mathcal{G}$ is mapped into the Universal Enveloping Algebra element $g^{\mathcal{F}} \in \mathcal{U}(\mathcal{G})$, given by

$$g^{\mathcal{F}} = \overline{f}^{\alpha}(g)\overline{f}_{\alpha}, \quad (\mathcal{F}^{-1} = \overline{f}^{\alpha} \otimes \overline{f}_{\alpha}).$$

Four twist-deformations: the jordanian deformations ($\epsilon = 0$) based on $\mathcal{F}$ (case *I*) and $\mathcal{F}_\tau$ (case *II*) and the extended jordanian deformations ($\epsilon = 1$) based on $\mathcal{F}$ (case *III*) and $\mathcal{F}_\tau$ (case *IV*).

Basis of twist-deformed generators (case *I*):

$$\begin{aligned}
P_+^{\mathcal{F}} &= P_+\frac{1}{1+\rho P_+},\\
P_-^{\mathcal{F}} &= P_-(1+\rho P_+),\\
P_j^{\mathcal{F}} &= P_j,\\
M^{\mathcal{F}} &= M,\\
M_{+j}^{\mathcal{F}} &= M_{+j}\frac{1}{1+\rho P_+},\\
M_{-j}^{\mathcal{F}} &= M_{-j}(1+\rho P_+),\\
N^{\mathcal{F}} &= N
\end{aligned}$$

$$\begin{aligned}
x_+^{\mathcal{F}} &= x_+\frac{1}{1+\rho P_+},\\
x_-^{\mathcal{F}} &= x_-(1+\rho P_+),\\
x_j^{\mathcal{F}} &= x_j.
\end{aligned}$$

The undeformed generators can be expressed in terms of the deformed generators on the basis of inverse formulas.

$\mathcal{F}_\tau$ twist-generator basis:

$$
\begin{aligned}
P^{\mathcal{F}}_{\bullet} &= P_{\bullet}, \\
M^{\mathcal{F}} &= \frac{1+2\rho P_+}{1+\rho P_+} M + \epsilon \left(\frac{\rho P_j}{1+\rho P_+} - \ln(1+\rho P_+) \frac{P_j}{P_+} \right) M, \\
M^{\mathcal{F}}_{+j} &= M_{+j} + \epsilon \ln(1+\rho P_+) M_{+j}, \\
M^{\mathcal{F}}_{-j} &= M_{-j} + \frac{2\rho P_j}{1+\rho P_+} M + \\
&\quad \epsilon \left(\ln(1+\rho P_+) \frac{P_j}{P_+} \delta_{jk} + \frac{2\rho P_j P_k}{(1+\rho P_+) P_+} - 2 \ln(1+\rho P_+) \frac{P_j P_k}{{P_+}^2} \right) \\
N^{\mathcal{F}} &= N - \epsilon \epsilon_{jk} \ln(1+\rho P_+) \frac{P_k}{P_+} M_{+j}
\end{aligned}
$$

$$
\begin{aligned}
x^{\mathcal{F}}_+ &= x_+, \\
x^{\mathcal{F}}_- &= x_- + \frac{2\rho\hbar}{1+\rho P_+} M + 2\epsilon\hbar \left(\frac{\rho}{1+\rho P_+} - \frac{\ln(1+\rho P_+)}{P_+} \right) \frac{P_j}{P_+} M_{+j}, \\
x^{\mathcal{F}}_j &= x_j - \epsilon \epsilon_{jk} \hbar \ln(1+\rho P_+) \frac{1}{P_+} M_{+j}.
\end{aligned}
$$

For an operator Ω the hermiticity condition is $\Omega^\dagger = \Omega$.
The pseudohermiticity condition is $\Omega^\dagger = \eta\Omega\eta^{-1}$, for an invertible hermitian operator $\eta = \eta^\dagger$.
Case *I* (jordanian $\mathcal{F}$ twist with $\epsilon = 0$):

$$\begin{array}{rcll} {P_\bullet^{\mathcal{F}}}^\dagger &=& \eta^\lambda P_\bullet^{\mathcal{F}} \eta^{-\lambda}, & \forall\lambda\in\mathbb{R}, \\ {M^{\mathcal{F}}}^\dagger &=& M^{\mathcal{F}}, & i.e.\ \lambda = 0, \\ {M_{+j}^{\mathcal{F}}}^\dagger &=& \eta^\lambda M_{+j}^{\mathcal{F}} \eta^{-\lambda}, & \forall\lambda\in\mathbb{R}, \\ {M_{-j}^{\mathcal{F}}}^\dagger &=& \eta M_{-j}^{\mathcal{F}} \eta^{-1}, & i.e.\ \lambda = 1, \\ {N^{\mathcal{F}}}^\dagger &=& \eta^\lambda N^{\mathcal{F}} \eta^{-\lambda}, & \forall\lambda\in\mathbb{R}, \end{array}$$

$$\begin{array}{rcll} {x_+^{\mathcal{F}}}^\dagger &=& \eta^\lambda x_+^{\mathcal{F}} \eta^{-\lambda}, & \forall\lambda\in\mathbb{R}, \\ {x_-^{\mathcal{F}}}^\dagger &=& \eta x_-^{\mathcal{F}} \eta^{-1}, & i.e.\ \lambda = 1, \\ {x_j^{\mathcal{F}}}^\dagger &=& \eta^\lambda x_j^{\mathcal{F}} \eta^{-\lambda}, & \forall\lambda\in\mathbb{R}, \end{array}$$

for the hermitian $\eta = 1 + \rho P_+$.
The deformed subalgebra contains a subalgebra of hermitian operators (for $\lambda = 0$). The operators $M_{-j}^{\mathcal{F}}, x_-^{\mathcal{F}}$ are not hermitian.

Case *II* (jordanian $\mathcal{F}_\tau$ twist with $\epsilon = 0$):

$$\begin{array}{rcl} P_\bullet^{\mathcal{F}\dagger} &=& \eta^\lambda P_\bullet^{\mathcal{F}} \eta^{-\lambda}, \quad \forall\lambda \in \mathbb{R}, \\ M^{\mathcal{F}\dagger} &=& \eta M^{\mathcal{F}} \eta^{-1}, \quad i.e.\ \lambda = 1, \\ M_{+j}^{\mathcal{F}\dagger} &=& \eta^\lambda M_{+j}^{\mathcal{F}} \eta^{-\lambda}, \quad \forall\lambda \in \mathbb{R}, \\ N^{\mathcal{F}\dagger} &=& \eta^\lambda N^{\mathcal{F}} \eta^{-\lambda}, \quad \forall\lambda \in \mathbb{R}, \end{array}$$

$$\begin{array}{rcl} x_+^{\mathcal{F}\dagger} &=& \eta^\lambda x_+^{\mathcal{F}} \eta^{-\lambda}, \quad \forall\lambda \in \mathbb{R}, \\ x_j^{\mathcal{F}\dagger} &=& \eta^\lambda x_j^{\mathcal{F}} \eta^{-\lambda}, \quad \forall\lambda \in \mathbb{R}, \end{array}$$

for the hermitian $\eta = \frac{1+\rho P_+}{1+2\rho P_+}$.
The deformed generators $M_{-j}^{\mathcal{F}}, x_-^{\mathcal{F}}$ do not have nice (pseudo)-hermiticity properties as the previous generators.
The deformed subalgebra contains a subalgebra of pseudo-hermitian operators for the common choice $\lambda = 1$.

Deformation as a non-linear W-algebra (case I):

$$\mathcal{P}_{drf} \quad : \quad \{P_\pm^{\mathcal{F}}, P_j^{\mathcal{F}}, M^{\mathcal{F}}, N^{\mathcal{F}}, M_{\pm j}^{\mathcal{F}}\},$$

$$\mathcal{P}_{obs} \quad : \quad \{P_\pm^{\mathcal{F}}, P_j^{\mathcal{F}}, M^{\mathcal{F}}, N^{\mathcal{F}}, M_{+j}^{\mathcal{F}}\}.$$

$$\begin{aligned}
[P_\pm^{\mathcal{F}}, M^{\mathcal{F}}] &= \pm i P_\pm^{\mathcal{F}} \mp i\rho P_+^{\mathcal{F}} P_\pm^{\mathcal{F}}, \\
[P_+^{\mathcal{F}}, M_{+j}^{\mathcal{F}}] &= 2iP_j^{\mathcal{F}} - 2i\rho P_+^{\mathcal{F}} P_j^{\mathcal{F}}, \\
[P_-^{\mathcal{F}}, M_{+j}^{\mathcal{F}}] &= 2iP_j^{\mathcal{F}}, \\
[P_-^{\mathcal{F}}, M_{-j}^{\mathcal{F}}] &= 2i\rho P_-^{\mathcal{F}} P_j^{\mathcal{F}}, \\
[P_j^{\mathcal{F}}, M_{\pm k}^{\mathcal{F}}] &= 2i\delta_{jk} P_\pm^{\mathcal{F}}, \\
[M^{\mathcal{F}}, M_{\pm j}^{\mathcal{F}}] &= \mp i M_{\pm j}^{\mathcal{F}} \pm i\rho M_{\pm j}^{\mathcal{F}} P_+^{\mathcal{F}}, \\
[N^{\mathcal{F}}, M_{\pm j}^{\mathcal{F}}] &= i\epsilon_{jk} M_{\pm k}^{\mathcal{F}}, \\
[M_{+j}^{\mathcal{F}}, M_{-k}^{\mathcal{F}}] &= -2i\delta_{jk} M^{\mathcal{F}} - 2i\epsilon_{jk} N^{\mathcal{F}} - 2i\rho M_{+j}^{\mathcal{F}} P_k^{\mathcal{F}}, \\
[M_{-j}^{\mathcal{F}}, M_{-k}^{\mathcal{F}}] &= 2i\rho (M_j^{\mathcal{F}} P_k^{\mathcal{F}} - M_{-k}^{\mathcal{F}} P_j^{\mathcal{F}}),
\end{aligned}$$

The second order (mass-term) Casimir of the Poincaré algebra

$$C_2 \;=\; P_+P_- - {P_1}^2 - {P_2}^2$$

remains undeformed under all four twist-deformations $I - IV$. In all cases

$$C_2 \;=\; P_+^{\mathcal{F}}P_-^{\mathcal{F}} - {P_1^{\mathcal{F}}}^2 - {P_2^{\mathcal{F}}}^2.$$

Due to their form, all ten twisted generators (collectively denoted as $g_I^{\mathcal{F}}$, $I = 1, \ldots, 10$) entering their respective deformed Poincaré algebras, commute with C_2:

$$[g_I^{\mathcal{F}}, C_2] \;=\; 0, \quad \forall g_I^{\mathcal{F}}.$$

Consequences of the deformations:
Let's set for simplicity the transverse momenta $P_j \equiv 0$.
In the undeformed case let's set

$$\begin{aligned} x &= P_+, \\ y &= P_-, \\ e &= P_0, \\ f &= P_1. \end{aligned}$$

Since $P_\pm = P_0 \pm P_1$, we have

$$\begin{aligned} e &= \frac{1}{2}(x+y), \\ f &= \frac{1}{2}(x-y), \end{aligned}$$

with e representing the energy of the system.
For a massive representation we get, on shell,

$$\begin{aligned} xy &= m^2, \\ e^2 - f^2 &= m^2. \end{aligned}$$

(without loss of generality $m = 1$).

Therefore

$$\begin{aligned} xy &= 1, \\ e^2 - f^2 &= 1, \\ y &= \frac{1}{x}, \\ e &= \frac{1}{2}(x + \frac{1}{x}), \\ f &= \frac{1}{2}(x - \frac{1}{x}). \end{aligned}$$

In order to have a positive energy e, x should be non-negative. The observables are therefore bounded in the domains

$$\begin{aligned} x &\in]0, +\infty], \\ y &\in]0, +\infty], \\ e &\in [1, +\infty], \\ f &\in [-\infty, +\infty]. \end{aligned}$$

The rest condition for the P_1 momentum corresponds to $f = 0$. It is obtained at $x = 1$. For this value the energy is minimal ($e = 1$).

Let us denote the deformed variables with a bar and the deformation parameter with "z":

$$\begin{aligned} \bar{x} &= \frac{x}{1+zx}, \\ \bar{y} &= y(1+zx). \end{aligned}$$

The condition

$$z \geq 0$$

has to be imposed to avoid singularities.
On-shell we have

$$\begin{aligned} \bar{y} &= \frac{1}{x}(1+zx), \\ \bar{e} &= \frac{1}{2}(\bar{x}+\bar{y}) = \frac{1}{2}\left(\frac{x}{1+zx} + \frac{1}{x}(1+zx)\right), \\ \bar{f} &= \frac{1}{2}(\bar{x}-\bar{y}) = \frac{1}{2}\left(\frac{x}{1+zx} - \frac{1}{x}(1+zx)\right). \end{aligned}$$

The rest condition $\bar{f} = 0$ for the deformed P_1 momentum is obtained for

$$x = \frac{1}{1-z}.$$

It can only be obtained for $z < 1$.
The range of the deformation parameter z is

$$0 \leq z < 1.$$

The range of the deformed operators is modified with respect to the range of the undeformed operators:

$$\begin{aligned} \bar{x} &\in]0, \frac{1}{z}[, \\ \bar{y} &\in]z, +\infty], \\ \bar{e} &\in [1, +\infty], \\ \bar{f} &\in [-\infty, \frac{1}{2}(z - \frac{1}{z})[. \end{aligned}$$

For the deformed energy we have $\lim_{\bar{x}\to 0^+} \bar{e} = +\infty$, while $\lim_{\bar{x}\to \frac{1}{z}^-} \bar{e} = \frac{1}{2}(z + \frac{1}{z})$.
$\frac{1}{z}$ can be interpreted as the maximal admissible P_+ momentum.

Induced by the coproduct, the 2-particle addition formula for the deformed P_+ momenta reads as follows

$$\overline{x}_{1+2} = \frac{x_1 + x_2}{1 + z(x_1 + x_2)}.$$

Closely expressed in terms of the deformed P_+ momenta we obtain the non-linear addition formula

$$\overline{x}_{1+2} = \frac{\overline{x}_1 + \overline{x}_2 - 2z\overline{x}_1\overline{x}_2}{1 - z^2\overline{x}_1\overline{x}_2}.$$

It is useful to compare this formula with the non-linear addition of velocities in special relativity. Let us change variables once more and set $\overline{x}_{1,2} = v_{1,2}$, $z = \frac{1}{c}$.
In special relativity we get

$$v_{1+2,s.r.} \quad = \quad \frac{v_1 + v_2}{1 + \frac{1}{c^2} v_1 v_2}.$$

In the above jordanian deformation we have

$$v_{1+2,jd.} \quad = \quad \frac{v_1 + v_2 - \frac{2}{c} v_1 v_2}{1 - \frac{1}{c^2} v_1 v_2}.$$

Let us compare their properties. Both non-linear addition formulas are symmetric in the $v_1 \leftrightarrow v_2$ exchange. They are both associative. They can also be defined in the box $0 \leq v_{1,2} \leq c$, so that the non-linear additive velocities belong to the $[0, c]$ range (in both cases if $v_1 = 0$, then $v_{1+2} = v_2$ and, if $v_1 = c$, $v_{1+2} = c$).
The main difference is that in special relativity the formula can be nicely extended to negative velocities belonging to the $-c \leq v_{1,2} \leq c$ box.

Thank you for the attention